HYPERBOLIC TYPE SPLINES FOR SOLVING STATIONARY DIFFUSION PROBLEMS IN 3-D DOMAIN WITH SPECIAL SOURCE FUNCTIONS

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Abstract. In the study of various spatial engineering problems (e.g. heat and mass transfer in multilayer media, diffusion and combustion processes), it is necessary to use 3-D partial differential equations (PDE), the solving of which is difficult. Therefore, in solving these problems, we apply the conservative averaging method. The conservative averaging method as an approximate analytical and numerical method for solving PDE or their systems with piece-wise constant (continuous) coefficients is under question. We consider averaging methods for solving the stationary 3-D boundary value problem of second order with piece-wise parameters in the 3-D domain for special source function. The hyperbolic-type splines, which interpolate middle integral values of a piece-wise smooth function, are considered. With the help of these splines, some boundary value problems of mathematical physics in 3-D with piece-wise coefficients are reduced to boundary value problems for ordinal differential equations in 1-D for one coordinate. The usage of this spline allows for diminishing the dimensions of the initial problem per one. The spline solution is used for different coordinates, in Cartesian coordinates, in cylindrical coordinates with axial symmetry and in spherical coordinates with axial symmetry. The analytical solution of the 1-D problem (boundary value problem for the ordinal differential equation) was compared with the corresponding spline function solution in the previously mentioned coordinates. Calculations to test theoretical assumptions and perform numerical experiments were proceeded with MATLAB.

Keywords: PDE, 3-D boundary value problem, conservative averaging method, 1-D initial value problem.

1. Introduction

Heat, mass transfer, and diffusion problem solutions in the 3-D domain are used by numerical and analytical methods. In [1], the three-dimensional hyperbolic and parabolic heat conduction equations with time-dependent, non-uniform distributed heat source is analytically solved in a finite solid cube. The straightforward solution is introduced for hyperbolic and parabolic conduction using eigen function. The Eigen function expansion method introduces the closed form solution of both Fourier and non-Fourier profiles. The paper [2] presents the 3D heat flow model for modelling heat transfer during diffusional phase transformations, which is based on LBM (Lattice Boltzmann method). This model considers the enthalpy of transformation. The work [3] proposed and solved inverse anomalous diffusion problems considering real data, a fractional hyperbolic advection-dispersion equation and differential evolution. To evaluate the direct problem, the extension of the classical finite difference method was proposed. In the inverse problem context, all the phenomenological models were able to obtain good estimates for the concentration profiles in both applications.

To improve study processes, various types of splines are widely used in modeling. In recent years, several new splines defined in nonpolynomial spaces have been proposed. C-B-splines were introduced in [4; 5]: A linearly parametrized set of curves, named C-curves, are an extension of cubic curves; they depend on a parameter $a > 0$, and their limiting case for $a \to 0$ is a cubic curve. C-B-splines are introduced as extensions of cubic uniform B-splines. Exponential B-splines have been studied in [6]: some results for exponential B-splines in tension are extended to higher order exponential B-splines. But these bases do not overlap in the cases of high order. The classification of UE-splines and the relationship of all kinds of existing splines are shown in [7]: the three types refer to polynomial, trigonometric and hyperbolic splines, which in this paper is unified and extended by a new kind of spline (UE-spline). Hyperbolic-polynomial splines, important in several applications, are a natural generalization of polynomial splines consisting of piecewise-defined functions with segments. Three-dimensional diffusion problems with discontinuous coefficients and one-dimensional Dirac sources are considered in [8]. The Dirac measure is, for example, a model of a loaded fiber or wire. Using superposition, the study covers the case of multiple line sources. Curvilinear sources can be encountered in various fields of science and various branches of engineering: solid and liquid mechanics [9; 10] is considered as a fractured porous medium that is studied at a scale such that the fractures can be modelled individually, models for flow in which the fractures are interfaces between subdomains are presented

[11]; in this paper is considered the coupling between two diffusion-reaction problems, one taking place in a three-dimensional domain Ω, the other in a one-dimensional subdomain Λ.

A computational methodology to simulate the diffusion of ions from point sources (e.g. ion channels) is described [12]. The outlined approach computes the ion concentration from a cluster of many ion channels at pre-specified locations as a function of time using the theory of propagation integrals. Modeling was done in 3 geometries – planar symmetry, cylindrical symmetry and spherical symmetry.

Often, solving these problems is complex and laborious. In addition, determining the accuracy of the results is problematic. Using special splines, it is possible to reduce the PDE problem to ODE.

A. Buikis in [13], 2017, gives a short history of conservative averaging method in the last 100 years. The main idea of CAM is that the new problem formulation in the main domain has fulfilled all energy peculiarities and fulfils conservation laws. CAM is considered with hyperbolic and parabolic functions. These methods were applied for the mathematical simulation of the mass transfer process in multi-layered underground systems, the mathematical foundation has been provided in the Doctor of sciences thesis 1987 [14].

With the help of created splines, the special 3-D problems of mathematical physics in one layer with piece-wise coefficients are reduced to problems for ODE in 1-D [15].

In this paper, we consider averaging and finite difference methods for solving the 3-D boundary value problem in the multi-layered domain. We consider the metals Fe and Ca concentration in the layered peat blocks. Using experimental data, the mathematical model for calculation of the concentration of metals in different points in peat layers is developed. A specific feature of these problems is that it is necessary to solve the 3-D boundary value problems for elliptic-type partial differential equations (PDEs) of second order with piece-wise diffusion coefficients in the layered domain. We develop here a finite-difference method for solving a problem of one, two and three peat blocks with periodical boundary conditions in the x direction. This procedure allows us to reduce the 3-D problem to a system of 2-D problems by using a circulant matrix.

The conservative averaging method (CAM) was developed as an approximate analytical and/or numerical method for solving a partial differential equation or its system with piece-wise constant (continuous) coefficients. The usage of this approximate method for separate relatively thin sub-domain or/and subdomain with a large heat conduction coefficient leads to a reduction of domain in which the solution must be found. To apply this method for all sub-domains of layered media, a special type of spline was constructed: the integral averaged values interpolating parabolic spline. The usage of this spline allows diminishing the dimensions of the initial problem per one. It is important that in all cases, the original problem with discontinuous coefficients from R_{n+1} transforms to a problem with continuous coefficients in R_n. These methods were applied for the mathematical simulation of the mass transfer process in multi-layered underground systems [16].

CAM as an approximate method for solution of some direct and inverse heat transfer problems is given in [17], here the solution is approximated with a polynomial: the conservative averaging method was developed as an approximate analytical and/or numerical method for solving partial differential equation or its system with piece-wise constant (continuous) coefficients. A method of conservative averaging for ill-posed inverse problems in some cases allows transforming them to well-posed inverse problems. Similarly, heat conduction problem for double layered ball is discussed [18]: heat conduction models for double layered spherical sample are developed. Parabolic (classic, based on Fourier's Law) and hyperbolic (based on Modified Fourier's Law) heat conduction equations are used to describe processes in the sample during Intensive Quenching. Solution and numerical results are obtained for the 1-D model using the conservative averaging method and transforming the original problem for a sphere to a new problem for a slab, with non-classic boundary condition. Models include boundary conditions of the third kind and non-linear BC case. Numerical results are presented for several relaxation time and initial heat flux values. CAM with special splines for solving of diffusion-convection problems with discontinuous coefficients for layered materials exposed to fire is shown in [19]: the temperature was calculated through the two layered material of gypsum products exposed to fire and the calculations were compared with the results obtained in the experiment at the Faculty of Environment of the Latvia University of Life Sciences and Technologies.

In [20] hyperbolic type splines are used for the solving heat and mass transfer 3-D problem in porous multi-layered axial symmetry domain: the approximation of the corresponding initial boundary value problem of the system of PDEs is based on the conservative averaging method (CAM) with special integral splines, this procedure allows reduce the 3-D axis-symmetrical transfer problem in multi-layered domain described by a system of PDEs to initial value problem for a system of ordinary differential equations (ODEs) of the first order. The special hyperbolic type approximation is used for solving the 3-D diffusion problem [21]: with the help of these splines, the initial boundary value problem (IBVP) concerning one coordinate is reduced to problems for a system of equations in the 2-D domain. This procedure also allows the reduction of the 2-D problem to a 1-D problem, and thus, the solution of the approximated problem can be obtained analytically. The accuracy of the approximated solution for the special 1-D IBVP is compared with the exact solution of the studied problem obtained with the Fourier series method.

In this paper, the special spline with two different functions, which interpolates middle integral values of piece-wise smooth function, is considered. Special hyperbolic and parabolic-type splines are developed for Cartesian, cylindrical and spherical coordinates. With the help of these splines, the special 3-D problems of mathematical physics in one layer with piece-wise coefficients transform to problems for ODE in 1-D. These splines contain parameters where they can be chosen for decreasing the error of the solution. In the limit case when for the hyperbolic spline the parameter tends to zero, we have the integral parabolic spline, obtained from A. Buikis.

2. Materials and methods

We have considered 3-D stationary diffusion problems with special source functions in different coordinates: Cartesian, cylindrical with axial symmetry and spherical coordinates with axial symmetry. We have obtained the analytical solutions of the corresponding boundary value problem for ODEs.

2.1. Formulation of problem in Cartesian coordinates

The process of diffusion is considered in 3-D parallelepiped

$$\Omega = \{(x,y,z): 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z\}.$$

We will consider the stationary 3-D problem of the linear diffusion theory. We will find the distribution of concentrations $c = c(x,y,z)$ in Ω at the point (x,y,z) by solving the following special 3-D boundary value problem for partial differential equation (PDE) with the source function (cosine-function) dependent on the x, y-directions:

$$\begin{cases} \frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + f_0 \cos\frac{\pi x}{2L_x}\cos\frac{\pi y}{2L_y} = 0 \\ \frac{\partial c(0,y,z)}{\partial x} = \frac{\partial c(x,0,z)}{\partial y} = 0, c(L_x,y,z) = 0, c\left(x,L_y,z\right) = 0 \\ D_z \frac{\partial c(x,y,0)}{\partial z} - \beta_z\left(c(x,y,0) - c_0 cos\frac{\pi x}{2L_x} cos\frac{\pi y}{2L_y}\right) = 0 \\ D_z \frac{\partial c(x,y,L_z)}{\partial z} + \alpha_z\left(c(x,y,L_z) - c_a cos\frac{\pi x}{2L_x} cos\frac{\pi y}{2L_y}\right) = 0 \end{cases} \quad (1)$$

where f_0, c_0, c_a – fixed constants;
D_x, D_y, D_z – constant diffusion coefficients;
α_z, β_z – constant mass transfer coefficients for the 3 kind boundary conditions.

We can obtain the analytical solution of (1) in the following form:

$$c(x,y,z) = g(z) cos\frac{\pi x}{2L_x} cos\frac{\pi y}{2L_y},$$

where the function $g(z)$ is the solution of boundary value problem for ODE

$$\begin{cases} g''(z) - a_0^2 g(z) + f_1 = 0, \\ g'(0) - \beta(g(0) - c_o) = 0,\ g'(L_z) + \alpha(g(L_z) - c_a) = 0 \end{cases} \quad (2)$$

where

$$f_1 = f_0/D_z, \beta = \beta_z/D_z, \alpha = \alpha_z/D_z,\ a_0^2 = \frac{\pi^2(D_y/L_y^2 + D_x/L_x^2)}{4D_z}.$$

We have following solution

$$g(z) = c_1 \sinh(a_0 z) + c_2 \cosh(a_0 z) + f_2,$$

where

$$c_1 = \beta/a_0(Cc_2 + f_2 - c_o), C_2 = \frac{(c_a - f_2)(\alpha/a_0 + \beta/a_0 c_3)}{\beta/a_0 c_3 + c_4}, f_2 = f_1/a_0^2,$$

$$c_3 = \cosh(a_0 L_z) + \alpha/a_0 \sinh(a_0 L_z), c_4 = \sinh(a_0 L_z) + \alpha/a_0 \cosh(a_0 L_z).$$

2.2. Problem in cylindrical coordinates with axial symmetry

The process of diffusion is considered in 3-D cylinder

$$\Omega = \{(r, z.\phi): 0 \le r \le R, 0 \le z \le L_z, 0 \le \phi \le 2\pi\}.$$

We will consider the stationary boundary value problem with axial symmetry.

We will find the distribution of concentrations $c = c(r,z)$ in Ω at the point (r,z) by solving the following special boundary value problem for partial differential equation (PDE) with the source function (cosine-function) dependent on the z-direction:

$$\begin{cases} D_r \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + f_0\, cos\frac{\pi z}{2L_z} = 0, \\ \frac{\partial c(r,0)}{\partial z} = \frac{\partial c(0,z)}{\partial r} = 0, c(r, L_z) = 0, \\ D_r \frac{\partial c(R,z)}{\partial r} + \alpha_r\left(c(R,z) - c_a \cos\frac{\pi z}{2L_z}\right) = 0, \end{cases} \quad (3)$$

where f_0, c_a – fixed constants;
D_r, D_z – constant diffusion coefficients;
α_r – constant mass transfer coefficient in the 3 kind boundary conditions.

We can obtain the analytical solution of (3) in the following form:

$$c(r,z) = g(r)\, cos\frac{\pi z}{2L_z},$$

where the function $g(r)$ is solution of boundary value problem for ODE:

$$\begin{cases} g''(r) + \frac{1}{r}g'(r) - a_0^2 g(r) + f_1 = 0, \\ g'(0) = 0, g'(R) + \alpha(g(R) - c_a) = 0, \end{cases} \quad (4)$$

where

$$f_1 = f_0/D_r, \alpha = \alpha_r/D_r\ a_0^2 = \frac{\pi^2 D_z/L_z^2}{4D_r}.$$

We have following solutions

$$g(r) = C_1 I_0(a_0 r) + f_2,$$

where

$$f_2 = \frac{f_1}{a_0^2}, C_1 = \frac{\alpha(c_a - f_2)}{a_0 I_1(a_0 R) + \alpha I_0(a_0 R)},$$

and I_0, I_1 – modified Bessel functions.

2.3. Problem in spherical coordinates with axial symmetry

The process of diffusion is considered in hemisphere

$$\Omega = \{(r,\theta,\phi): 0 \le r \le R, 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi\}.$$

We will consider the stationary boundary value problem with axial symmetry.

We will find the distribution of concentrations $c = c(r,\theta)$

in Ω at the point (r,θ) by solving the following special boundary value problem for partial differential equation (PDE) with the source function (cosine-function) dependent on the θ -direction:

$$\begin{cases} D_r \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial c}{\partial r}\right) + D_\theta \dfrac{1}{r^2 \sin(\theta)}\dfrac{\partial}{\partial \theta}\left(\sin(\theta)\dfrac{\partial c}{\partial \theta}\right) + f_0 \cos(\theta)/r^2 = 0 \\ \dfrac{\partial c(r,0)}{\partial \theta} = \dfrac{\partial c(0,\theta)}{\partial r} = 0, c(r,\pi/2) = 0 \\ D_r \dfrac{\partial c(R,\theta)}{\partial r} + \alpha_r(c(R,\theta) - c_a \cos(\theta)) = 0 \end{cases} \tag{5}$$

where f_0, c_a – fixed constants;
D_r, D_θ – constant diffusion coefficients,
α_r – constant mass transfer coefficient.

We can obtain (have obtained) the analytical solution of (5) in the following form:

$$c(r,\theta) = g(r)\cos(\theta),$$

where the function $g(r)$ is solution of boundary value problem for ODE:

$$\begin{cases} r^2 g''(r) + 2r g'(r) - a_0^2 g(r) + f_1 = 0, \\ g'(0) = 0, g'(R) + \alpha(g(R) - c_a) = 0, \end{cases} \tag{6}$$

where

$$f_1 = f_0/D_r, \alpha = \alpha_r/D_r, a_0^2 = \frac{2D_\theta}{D_r} > 2.$$

We have following solutions

$$g(r) = C_1 r^\gamma + f_2,$$

where

$$f_2 = \frac{f_1}{a_0^2}, \gamma = -0.5 + \sqrt{0.25 + a_0^2}, C_1 = \frac{\alpha(c_a - f_2)}{R^{\gamma-1}(\gamma + \alpha R)}.$$

3. Results and discussion

This chapter discusses the construction of hyperbolic splines (an approximate solution to a 1-D ordinary differential equation boundary value problem) based on the conservative averaging method (CAM).

With the help of the created splines, the 1-D boundary value problems found in the previous chapter were solved in three cases – in a parallelepiped (Cartesian coordinates), in a cylinder (cylindrical coordinates) and in a hemisphere (spherical coordinates). Spline components – functions contain a special parameter a, the numerical value of which determines the accuracy of the calculations of the entire spline method – the reduction of the calculation error.

When creating a spline in the case of Cartesian coordinates, the optimal value of the parameter a was calculated analytically (directly with the help of a formula), using the ordinary differential equation to which the original 3-D boundary value problem was reduced.

In the case of cylindrical and spherical coordinates, a variable - the correction coefficient (kor) – was additionally involved in the error reduction process to calculate the numerical value of the parameter a.

Applying of the parabolic spline, obtained from A. Buikis, the spline functions' parameter a has been chosen equal to 0.00001.

3.1. Averaging method in z-direction using integral spline in Cartesian coordinates

The conservative averaging method (CAM) is applied to solve the boundary value problem (BVP) (2) using the approximate solution – spline. BVP (2) differential equation is integrated for the variable z in the range from 0 to L_z, dividing by L_z. In the differential equation, instead of the function $g(z)$, a spline function $g(z)$ with two parametric functions f_{z1}, f_{z2} (see below) is inserted. The boundary conditions of BVP (2) are applied, and the form (7) is obtained, from which g_a is calculated. By inserting the calculated value g_a into the spline formula, the approximate solution of BVP (2) is obtained.

$$g(z) = g_a + mf_{z1}(z - L_z/2) + ef_{z2}(z - L_z/2),$$

where

$$g_a = \frac{1}{L_z}\int_0^{L_z} g(z)dz$$

is the averaged value,

$$\int_0^{L_z} f_{z1}dz = \int_0^{L_z} f_{z2}dz = 0,$$

$$f_{z1} = \frac{0.5L_z\, sinh\big(a(z-0.5L_z)\big)}{sinh(0.5aL_z)},\ f_{z2} = \frac{cosh\big(a(z-0.5L_z)\big)-A_0}{8\,sinh^2(0.25aL_z)}\ A_{0z} = \frac{sinh(0.5aL_z)}{0.5aL_z},$$

and $a = a_0$ is the optimal parameter.

We can see that the parameter a tends to zero, then the limit is the integral parabolic spline [16], because of

$$A_0 \to \frac{1}{12}\colon f_{z1} \to (z - L_z/2),\ f_{z2} \to \frac{(z-L_z/2)^2}{{L_z}^2} - \frac{1}{12}.$$

The unknown coefficients m, e we can determine from the boundary conditions of (2):

1. for $z = 0, md - ek - \beta(g_a - 0.5mL_z + eb - c_o) = 0,$
2. for $z = L_z, md + ek + \alpha(g_a + 0.5mL_z + eb - c_a) = 0,$

where

$$d = (aL_z/2)\coth(aL_z/2),\ k = (a/4)\coth(aL_z/4),\ b = \frac{(cosh(aL_z/2))-A_0}{8sinh^2(aL_z/4)}.$$

Therefore

$$e = g_e g_a + a_e, m = g_a g_m + a_m,\ g_e = (a_{11}\alpha + a_{21}\beta)/det,\ g_m = (a_{22}\beta - a_{12}\alpha)/det,$$

$$a_e = (c_o a_{21}\beta + c_a a_{11}\alpha)/det,\ a_m = (c_a a_{12}\alpha - c_o a_{22}\beta)/det,\ det = a_{11}a_{22} + a_{12}a_{21},$$

$$a_{11} = d + \beta L_z/2, a_{12} = k + b\beta, a_{21} = d + \alpha L_z/2\,, a_{22} = k + b\alpha.$$

Now the boundary value problem (2) is in the form

$$1/L_z\big(g'(L_z) - g'(0)\big) - a_0^2 g_a + f_1 = 0,\ g'(L_z) - g'(0) = 2ek \quad (7)$$

or

$$g_a = \frac{f_1 L_z + 2ka_e}{2kg_e + a_0^2 L_z}.$$

Example 1

We consider the following parameters for solving the boundary value problem (7):

$$f_0 = 0.1, L_z = 1, L_x = 1, L_y = 1,\ D_x = 10^{-2},\ D_y = 10^{-3},\ D_z = 10^{-4},$$

$$\alpha_z = 0.3,\ \beta_z = 0.1,\ c_0 = 5,$$

$$c_a = 2,\ a_0 = 16.4747.$$

The maximal errors (er) of the solutions of the parabolic and hyperbolic spline methods are compared, here the maximal error - the maximal difference (calculated by absolute value) between the obtained spline method solution and the analytical (exact) solution of the corresponding 1-D boundary value problem.

We have the following maximal errors: for hyperbolic spline $er_h = 2 \cdot 10^{-9}$, for parabolic spline $er_p = 0.9653$ (see $g(z)$ in Fig. 1 and $c(x, 0, z)$ in Fig. 2).

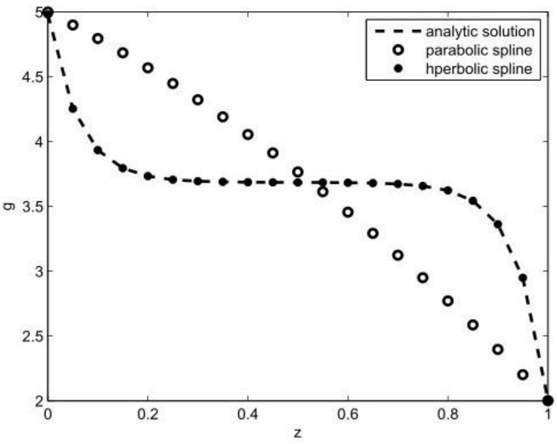

Fig. 1. **Solution $g(z)$, $er_h = 2 \cdot 10^{-9}$, $er_p = 0.9653$**

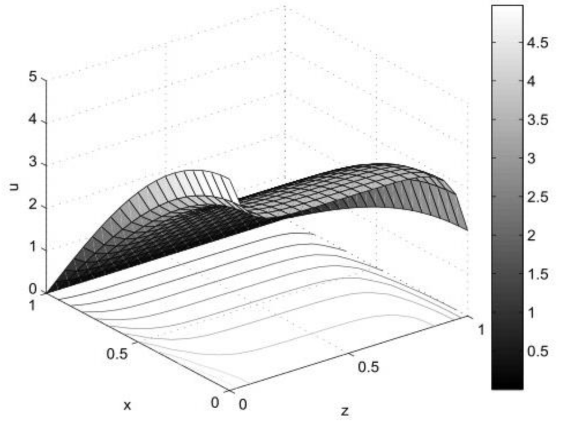

Fig. 2. **Solution $u = c(x, 0, z)$ for hyperbolic spline**

3.2. Averaging method in r-direction using integral spline in cylindrical coordinates

Applying CAM for solving (4) similarly Cartesian coordinates case (3.1) we use spline $g(r)$ with two fixed parametrical functions f_{r1}, f_{r2}

$$g(r) = g_a + m f_{r1}(r - R/2) + e f_{r2}(r - R/2),$$

where

$$g_a = \frac{1}{R^2}\int_0^R r g(r) dr \text{ is the averaged value, } \int_0^R r f_{r1} dr = \int_0^R r f_{r2} dr = 0,$$

$$f_{r1} = \frac{R^2 a^2 \sinh(a(r - 0.5R))}{4\sinh(0.5aR)}, f_{r2} = \frac{\cosh(a(r - 0.5R)) - A_0}{8\sinh^2(0.25aR)}, A_0 = \frac{\sinh(0.5aR)}{0.5aR},$$

$$d = Ra/2 \coth(aR/2), a = a_0 + kor,$$

where kor – correction parameter.

Correction parameter (kor) – the minimal difference (calculated by the absolute value) between the maximal errors of two adjacent spline method solutions (which differ from each other by a sufficiently small change in the correction coefficient).

The formula $a = a_0 + kor$ was used to obtain the minimal error of calculations, a - parameter of the approximate solution of the spline method, a_0 – constant, depending on the parameters of the respective boundary-value problem.

We can see if the parameter a tends to zero, then the limit is the integral parabolic spline.

The unknown coefficients m, e we can determine from boundary conditions (4):

1. for $r = 0, md_r - ek = 0$,
2. for $r = R, md_r + ek + \alpha(g_a + mb_m + eb_e - c_a) = 0$,

where

$$k = (a/4)\,coth(aR/4), d_r = 0.5 \cdot Rda^2/(d-1), b_e = \frac{(cosh(R/2)) - A_0}{8sinh^2(aR/4)}, b_m = \frac{R^2a^2}{4(d-1)}.$$

Therefore

$$e = (c_a - g_a)/g_1, m = ek/d_r, g_1 = 2k/\alpha + b_m k/d_r + b_e.$$

Now the boundary value problem (4) is in the form

$$2/R^2\left(Rg'(R)\right) - a_0^2 g_a + f_1 = 0, g'(R) = 2e, \quad (8)$$

or

$$g_a = \frac{f_1 g_1 R + 4kc_a}{4k + a_0^2 g_1 R}.$$

Example 2

We consider the following parameters for solving the boundary value problem (8):

$L_z = R = 1, f_0 = -0.1, \alpha_r = 10.01, c_a = 10, D_r = 10^{-2}, D_z = 10^{-2}, a_0 = 1.5708.$

Here – in the case of cylindrical coordinates (and also in the case of spherical coordinates, see Example 3), comparing spline methods, along with the maximal error, the corresponding correction coefficient (kor) has also been calculated.

We have following maximal errors: for hyperbolic spline $er_h = 0.00084, kor = -0.202$ (for $kor = -0.201$, $er_h = 0.00064$, but for $kor = -0.200$, $er_h = 0.0016$) for parabolic $er_p = 0.7476$ (see $g(r)$ in Fig. 3 and $c(r,z)$ in Fig. 4). If $\alpha_r = 0.01$ then $er_h = 0.00032$, $er_p = 0.3412$, but for $f_0 = 1.01$: $er_h = 0.00079, er_p = 0.8412$.

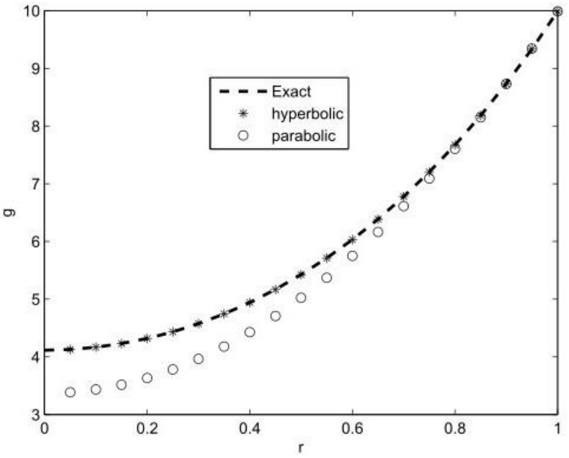

Fig. 3. **Solution $g(r)$, $er_h = 0.00084$ $er_p = 0.7476$**

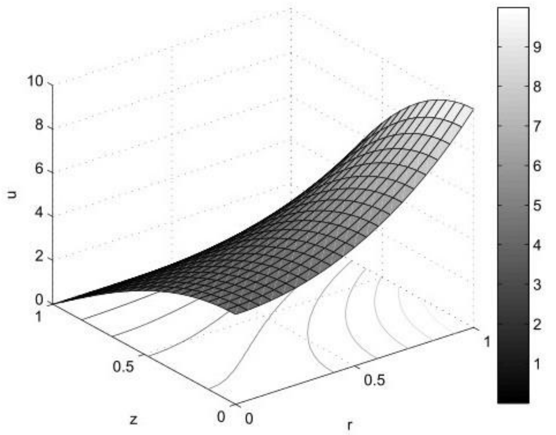

Fig. 4. **Solution $u = c(r,z)$ for hyperbolic spline**

3.3. Averaging method in r-direction using integral spline in spherical coordinates

Applying CAM for solving (6) similarly cylindrical coordinates case (3.2) we use spline $g(r)$ with two fixed parametrical functions f_{r1}, f_{r2}

$$g(r) = g_a + mf_{r1}(r - R/2) + ef_{r2}(r - R/2),$$

where

$g_a = \frac{1}{R^2}\int_0^R rg(r)dr$ is the averaged value, $\int_0^R rf_{r1}dr = \int_0^R rf_{r2}dr = 0,$

$$f_{r1} = \frac{0.5R\,sinh\left(a(r-0.5R)\right)}{sinh(0.5aR)}, f_{r2} = \frac{cosh\left(a(r-0.5R)\right)-A_0}{8\,sinh^2(0.25aR)}, A_0 = \frac{sinh(0.5aR)}{0.5aR},$$

$$a = a_0 + kor,$$

where kor – the correction parameter.

We can see if the parameter a tends to zero, then the limit is the integral parabolic spline [16].

The unknown coefficients m, e we can determine from boundary conditions (6):

1. for $r = 0, md - ek = 0$,
2. for $r = R, md + ek + \alpha(g_a + mR/2 + eb_e - c_a) = 0$,

where

$$k = (a/4)\,coth(aR/4),\ \mathrm{d} = 0.5 \cdot Racoth(aR/2),\ b = \frac{(cosh(R/2)) - A_0}{8sinh^2(aR/4)}.$$

Therefore

$$e = (c_a - g_a)/g_1,\ m = ek/d,\ g_1 = 2k/\alpha + b + 0.5kR/d.$$

Now the boundary value problem (6) is in the form

$$1/R\left(R^2 g'(R)\right) - a_0^2 g_a + f_1 = 0, g'(R) = 2ek \tag{9}$$

or

$$g_a = \frac{f_1 g_1 R + 2kRc_a}{2Rk + a_0^2 g_1}.$$

Example 3

We consider the following parameters for solving the boundary value problem (9):

$$R = 2,\ f_0 = 0.1, \alpha_r = 0.003,\ c_a = 1,\ D_t = 10^{-2},\ D_r = 10^{-4},\ a_0 = 14.142.$$

We have obtained the following maximal errors with their corresponding correction parameters:

for hyperbolic spline $er_h = 0.0350$, $kor = -7.0$ (for $kor = -7.005$, $er_h = 0.0351$, bat for $kor = -7.01$, $er_h = 0.0352$), for parabolic $er_p = 1.968$. (see $g(r)$ in Fig. 5 and $c(r, \theta)$ in Fig. 6). If $R = 1$ then $er_h = 0.0306$, $kor = .0$, (for $kor = -0.1$, $er_h = 0.0310$\$, bat for $kor = 0.01$, $er_h = 0.0311$) $er_p = 1.9664$.

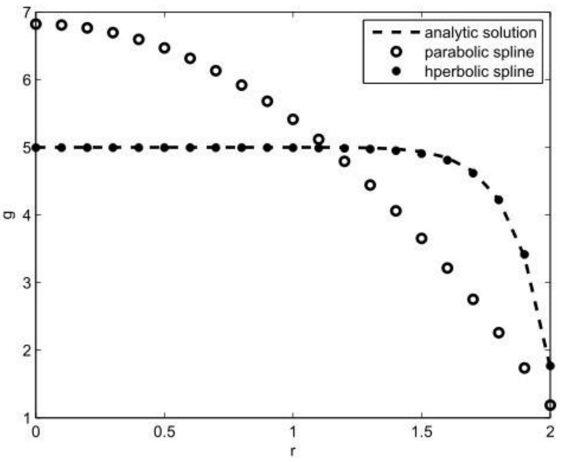

Fig. 5. **Solution $g(r)$, $er_h = 0.0350\ er_p = 1.968$**

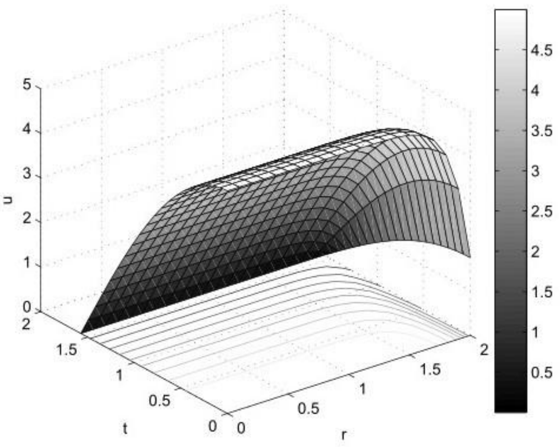

Fig. 6. **Solution $u = c(r, \theta)$ for hyperbolic spline**

4. Conclusions

1. The article discusses a 3-D stationary diffusion boundary value problem (BVP) with a special source function in various three-dimensional domains – a 3-D parallelepiped, a 3-D cylinder, and a 3-D hemisphere. The 3-D BVP in each of the mentioned domains is reduced to a 1-D ordinary differential equation boundary value problem in three cases – a parallelepiped (in Cartesian coordinates), a cylinder (in cylindrical coordinates) and a hemisphere (in spherical coordinates). Based on the conservative averaging method (CAM), the hyperbolic spline method was developed for solving the above-mentioned 1-D BVP in three cases – a parallelepiped, a cylinder and a hemisphere, considering the above-mentioned coordinates. Analytical solutions of the relevant 1-D boundary value problems (in Cartesian, cylindrical and spherical coordinates) were also obtained

to assess the accuracy of the developed hyperbolic spline method, as well as the accuracy of the parabolic spline method used in the calculations.

2. The most important thing in determining the calculation accuracy of the hyperbolic spline method in all three cases (Cartesian, cylindrical and spherical coordinates) was the calculation of the spline function parameter a. In the case of Cartesian coordinates, it could be calculated analytically, while in the case of cylindrical and spherical coordinates the parameter was calculated by introducing (defining) an additional element in the calculation process – the correction parameter kor.
3. Higher calculation accuracy with hyperbolic splines was achieved in the case of Cartesian coordinates, for example, (er_h- maximal error for hyperbolic splines, er_p – parabolic). The above comparison and other numerical experiments show the advantage of the hyperbolic spline method over parabolic splines. In the case of cylindrical and spherical coordinates, the accuracy of calculations is lower, for example, in cylindrical coordinates – $er_h = 0.00084$, $kor = -0.202$ (kor – correction parameter), $er_p = 0.7476$; in spherical coordinates – $er_h = 0.0350$, $kor = -7.0$, $er_p = 1.968$. Lower accuracy indicators in the case of cylindrical and spherical coordinates indicate a higher degree of complexity of the problem to be solved – first of all, this is attributable to the configuration of the 3-D domain, then – additional steps must be taken in the execution of the algorithm (for example, including the correction parameter kor in the solution process).
4. Despite the lower calculation accuracy obtained in the case of cylindrical and spherical coordinates, it is necessary to consider the more diverse spheres and possibilities of their application, therefore the created hyperbolic spline mathematical model is fully usable in the current situation, at the same time it can also be considered as an object of further research with the aim of its improvement.

Author contributions

The contribution of all three authors to the development of a given publication is equivalent. All authors have read and agreed to the published version of the manuscript.

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